

PLURIPOTENTIAL ENERGY AND LARGE DEVIATION

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ABSTRACT. We generalize results from [13] relating pluripotential energy with the electrostatic energy of a measure given in [5]. As a consequence, we obtain a large deviation principle for a canonical sequence of probability measures on a nonpluripolar compact set $K \subset \mathbb{C}^n$. This is a special case of a result of R. Berman [3]. For $n = 1$, we include a proof that uses only standard techniques of weighted potential theory.

1. Introduction.

In [5], Berman, Boucksom, Guedj and Zeriahi defined the notion of *electrostatic energy* $E^*(\mu)$ associated with a probability measure μ on a compact Kähler manifold X of dimension n . In [13], after specializing to the case of a compact set $K \subset \mathbb{C}^n \subset \mathbb{P}^n$ we studied two related functionals $J(\mu)$, $W(\mu)$ and their weighted counterparts $J^Q(\mu)$, $W^Q(\mu)$. The functionals involve weak-* approximations of μ by discrete measures. The relation is given as Corollaries 5.7 and 5.8 in [13]. They give another interpretation of $E^*(\mu)$ which we called *pluripotential energy*.

Our main goals in this paper are Theorems 4.4, 4.6 and 5.1:

- (1) Theorems 4.4 and 4.6 sharpen and clarify the relation between E^* (Definition 2.2) and the J , J^Q and W , W^Q functionals (Definitions 4.1 and 4.2) for measures with compact support $K \subset \mathbb{C}^n$. In [13] these functionals were defined using a compact, convex set $H \supset K$, although their values were independent of the choice of H . In this paper the functionals are defined directly in terms of K ; moreover K can be any nonpluripolar compact set.

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- (2) The functionals J, J^Q and W, W^Q can be defined using either a “limsup” or a “liminf” and the equality of these two, which we demonstrate in Theorems 4.4 and 4.6, leads immediately to a large deviation principle (LDP) in Theorem 5.1 *with rate function given in terms of these functionals*. In this setting there is a second approach to a LDP using a Legendre transform which leads to the E^* functional; hence the relation between E^* and the J, J^Q and W, W^Q functionals may be deduced from the *uniqueness* of the rate function (see Remark 5.7).
- (3) A key ingredient in the proofs is a deep result of Berman, Boucksom and Nystrom [6] on the convergence of the empirical measures of Fekete points, restated here as Theorem 2.4. The transition from an energy of measures to discrete approximations of this energy yielding a LDP – without utilizing a Legendre transform – can be achieved in other potential-theoretic situations when one has the analogue of Theorem 2.4, cf., [8] for Angelsco ensembles. The authors are preparing a subsequent work on a LDP in a more general “vector energy” setting in the univariate case.
- (4) The utilization of a *strong Bernstein-Markov measure* (Definition 3.7) is crucial for either approach to the LDP. In Corollary 3.5 we show that *any* compact set admits a Bernstein-Markov measure and thus (Corollary 3.8) any *nonpluripolar* compact set admits a strong Bernstein-Markov measure.

Theorem 5.1 states a LDP for a canonical sequence $\{\sigma_k\}$ of measures on $\mathcal{M}(K)$, the space of probability measures on K . The measures σ_k , which form a determinantal process, are the push-forwards of measures on products of K and they are canonical in the sense that in the univariate case, i.e., $K \subset \mathbb{C}$, they include the joint probability distributions of the eigenvalues of ensembles of unitary matrices restricted to K . Robert Berman [3] has given two proofs of such a LDP in a more general context. In this paper we also include two proofs. Our first proof is similar in spirit to Berman’s first proof in the sense that one establishes an “easy” upper bound and then a less-trivial lower bound. Here the effort is concealed in Theorems 4.4 and 4.6. Precisely, Theorem 4.4 shows the equivalence of the appropriate form of $E^*(\mu)$ and an L^∞ -type discretization $W(\mu)$ for $\mu \in \mathcal{M}(K)$; the equivalence with the L^2 -type discretization $J(\mu)$ follows for a J functional defined with

a Bernstein-Markov measure ν . From this latter equivalence the LDP is obtained if ν is a strong Bernstein-Markov measure using a standard result (Theorem 4.1.11 [14]). As mentioned in item (3) above, this procedure can be exploited in other settings. Indeed, in section 6 we indicate how in the univariate setting our approach relies solely on standard techniques and results from weighted potential theory.

The second proof is the same as in Berman's work and was described to us by Sébastien Boucksom. It relies on a result in [14] which states under general conditions that there is a LDP if a certain functional is Gateau differentiable; moreover the rate function is given as a Legendre transform. *In this setting the rate function is given directly in terms of E^* .* The differentiability follows from a fundamental result of Berman and Boucksom [4] restated here as Theorem 2.1. Indeed, in the multivariate setting, Theorem 2.4 is ultimately a consequence of Theorem 2.1.

Ben Arous and Guionnet [2], building on work of Voiculescu, first established an LDP for the Gaussian unitary ensemble (GUE). Their method was extended to more general ensembles by Hiai and Petz [16] and other authors. In the univariate case, the method utilized in each of the two proofs of the LDP on compact sets presented here are different from that utilized in [2]. Furthermore, given that there exist strong Bernstein-Markov measures which are not absolutely continuous with respect to Lebesgue measure (see Remark 3.6 and Corollary 3.8), these methods give a LDP in the univariate case not included in [2] or [16]. Berman [3] has proved a LDP on closed and unbounded sets by modifying his second proof in the compact case. This includes the GUE setting (see Remark 6.4).

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2. Preliminaries.

In this section, we summarize important results in [4], [5] and [6] adapted to our \mathbb{C}^n setting. We write $L(\mathbb{C}^n)$ for the set of all plurisubharmonic (psh) functions u on \mathbb{C}^n with the property that $u(z) - \log |z|$ is bounded above as $|z| \rightarrow \infty$ and

$$L^+(\mathbb{C}^n) = \{u \in L(\mathbb{C}^n) : u(z) \geq \log^+ |z| + C\}$$

where C is a constant depending on u . For locally bounded psh functions, e.g., for $u \in L^+(\mathbb{C}^n)$, the complex Monge-Ampère operator $(dd^c u)^n$ is well-defined as a positive measure. Here we normalize our definition of $dd^c = \frac{i}{\pi} \partial \bar{\partial}$ so that

$$\int_{\mathbb{C}^n} (dd^c u)^n = 1 \text{ for all } u \in L^+(\mathbb{C}^n).$$

For arbitrary $u \in L(\mathbb{C}^n)$ one can define the weak-* limit

$$NP(dd^c u)^n := \lim_{j \rightarrow \infty} (\mathbf{1}_{\{u > -j\}} \cdot (dd^c \max[u, -j])^n)$$

(cf., [1]). For $u \in L(\mathbb{C}^n)$ with

$$(2.1) \quad \int_{\mathbb{C}^n} NP(dd^c u)^n = 1,$$

we write $(dd^c u)^n := NP(dd^c u)^n$. As in [13], we only consider $u \in L(\mathbb{C}^n)$ satisfying (2.1). For such u , $(dd^c u)^n$ puts no mass on pluripolar sets.

Let $K \subset \mathbb{C}^n$ be compact – an assumption remaining in force throughout the paper – and let Q be a lowersemicontinuous function with $\{z \in K : e^{-Q(z)} > 0\}$ nonpluripolar (note that this implies K be nonpluripolar). We denote the collection of such Q as $\mathcal{A}(K)$ and we call Q an *admissible weight*. We define the *weighted pluricomplex Green function* $V_{K,Q}^*(z) := \limsup_{\zeta \rightarrow z} V_{K,Q}(\zeta)$ where

$$V_{K,Q}(z) := \sup\{u(z) : u \in L(\mathbb{C}^n), u \leq Q \text{ on } K\}.$$

The case $Q \equiv 0$ is the “unweighted” case and we simply write V_K . We have $V_{K,Q}^* \in L^+(\mathbb{C}^n)$ and we call the measure

$$\mu_{K,Q} := (dd^c V_{K,Q}^*)^n,$$

which has support in K , the *weighted equilibrium measure*; if $Q \equiv 0$ we write $\mu_K = (dd^c V_K^*)^n$. An example is the n -torus $T = \{z \in \mathbb{C}^n : |z_j| = 1, j = 1, \dots, n\}$ where

$$V_T(z) = \max_j [\log |z_j|, 0] \text{ and } \mu_T = \frac{1}{2\pi} d\theta \times \dots \times \frac{1}{2\pi} d\theta.$$

We say K is locally regular if for each $z \in K$ the unweighted pluricomplex Green function for the set $K \cap \overline{B}(z, r)$ is continuous for $r = r(z) > 0$ sufficiently small. Here $B(z, r)$ denotes the Euclidean ball with center z and radius r . We say K is L -regular if $V_K = V_K^*$; i.e., V_K is continuous.

If K is locally regular and Q is continuous, then $V_{K,Q}$ is continuous. In general, it is known that

$$(2.2) \quad \text{supp}(\mu_{K,Q}) \subset \{z \in K : V_{K,Q}^*(z) \geq Q(z)\}$$

and that $V_{K,Q}^* = Q$ on $\text{supp}(\mu_{K,Q})$ q.e., i.e., except perhaps for a pluripolar set.

The strictly psh function $u_0(z) := \frac{1}{2} \log(1 + |z|^2)$ belongs to the class $L^+(\mathbb{C}^n)$. For $u \in L^+(\mathbb{C}^n)$ define

$$(2.3) \quad E(u) := \frac{1}{n+1} \int_{\mathbb{C}^n} \sum_{j=0}^n (u - u_0) (dd^c u)^j \wedge (dd^c u_0)^{n-j}.$$

The functional E is a primitive for the complex Monge-Ampère operator (see Propositions 4.1 and 4.4 of [4]).

For $Q \in \mathcal{A}(K)$, define

$$P(Q) := V_{K,Q}^*.$$

The composition of the E and P operators is Gateaux differentiable; this non-obvious result (Theorem 2.1) was proved by Berman and Boucksom in [4]. It is the key ingredient in proving Theorem 2.4.

Theorem 2.1. *The functional $E \circ P$ is Gateaux differentiable; i.e., for $K \subset \mathbb{C}^n$ nonpluripolar and $Q \in \mathcal{A}(K)$, $F(t) := (E \circ P)(Q + tv)$ is differentiable for all $v \in C(K)$ and $t \in \mathbb{R}$. Furthermore,*

$$F'(0) = \int_K v (dd^c P(Q))^n.$$

Next, for $k = 1, 2, \dots$ let \mathcal{P}_k denote the space of holomorphic polynomials of degree at most k . We let $\dim \mathcal{P}_k = N_k = \binom{n+k}{k}$ and $\sum_{j=1}^{N_k} \deg(e_j) = \frac{n}{n+1} k N_k$ where $\{e_1, \dots, e_{N_k}\}$ is the standard monomial basis for \mathcal{P}_k . The weighted k -th order diameter of K with $Q \in \mathcal{A}(K)$ is

$$\delta^{Q,k}(K) := \left(\max_{x_1, \dots, x_{N_k} \in K} |VDM_k(x_1, \dots, x_{N_k})| e^{-kQ(x_1)} \dots e^{-kQ(x_{N_k})} \right)^{\frac{n+1}{nkN_k}}$$

where

$$(2.4) \quad VDM_k(x_1, \dots, x_{N_k}) = \det[e_i(x_j)]_{i,j=1, \dots, N_k}.$$

We introduce the shorthand notation

$$VDM_k^Q(x_1, \dots, x_{N_k}) := VDM_k(x_1, \dots, x_{N_k}) e^{-kQ(x_1)} \dots e^{-kQ(x_{N_k})}.$$

The limit

$$(2.5) \quad \lim_{k \rightarrow \infty} (\delta^{Q,k}(K))^{\frac{n}{n+1}} =: \bar{\delta}^Q(K)$$

exists (cf., [4] or [12]) and is called the (normalized) weighted transfinite diameter of K with $Q \in \mathcal{A}(K)$; moreover $\bar{\delta}^Q(K) > 0$ since Q is admissible. The nonstandard $\frac{n}{n+1}$ in (2.5) is to achieve a nicer *Rumely-type formula* ([4], p. 383 and see [19]):

$$(2.6) \quad -\log \bar{\delta}^Q(K) = E(V_{K,Q}^*) - E(V_T).$$

Definition 2.2. The *electrostatic* or *pluripotential energy* $E^*(\mu)$ of a measure $\mu \in \mathcal{M}(K)$ from [5] is a Legendre-type transform of the functional E :

$$(2.7) \quad E^*(\mu) = \sup_{Q \in C(K)} [E(V_{K,Q}^*) - \int_K Q d\mu] + \int_K u_0 d\mu$$

(see also Proposition 5.6 in [13]).

In [5] it is shown that $\mu_{K,Q}$ minimizes the functional

$$(2.8) \quad E^*(\mu) + \int_K (Q - u_0) d\mu$$

over all $\mu \in \mathcal{M}(K)$; the minimal value is $E(V_{K,Q}^*)$. This gives a remarkable generalization of the univariate setting (see Remark 4.5 and section 6). Moreover, in [5], a variational approach to the following special case of a result of Guedj and Zeriahi [15] was given.

Theorem 2.3. *Let $\mu \in \mathcal{M}(K)$ with $E^*(\mu) < +\infty$. Then there exists $u \in L(\mathbb{C}^n)$ with $(dd^c u)^n = \mu$ and $\int_K u d\mu > -\infty$.*

Finally, we recall a crucial result from [6] on “asymptotic weighted Fekete arrays.”

Theorem 2.4. *Let $K \subset \mathbb{C}^n$ be nonpluripolar and let $Q \in \mathcal{A}(K)$. For each k , take N_k points $x_1^{(k)}, \dots, x_{N_k}^{(k)} \in K$ for which*

$$\lim_{k \rightarrow \infty} (|VDM_k^Q(x_1^{(k)}, \dots, x_{N_k}^{(k)})|)^{\frac{1}{kN_k}} = \bar{\delta}^Q(K).$$

*Then $\nu_k := \frac{1}{N_k} \sum_{j=1}^{N_k} \delta_{x_j^{(k)}} \rightarrow \mu_{K,Q}$ weak- * .*

3. Bernstein-Markov Property.

Given a compact set $K \subset \mathbb{C}^n$ and a measure ν on K , we say that (K, ν) satisfies a Bernstein-Markov property if for all $p_k \in \mathcal{P}_k$,

$$\|p_k\|_K := \sup_{z \in K} |p_k(z)| \leq M_k \|p_k\|_{L^2(\nu)} \text{ with } \limsup_{k \rightarrow \infty} M_k^{1/k} = 1.$$

More generally, for $K \subset \mathbb{C}^n$ compact, $Q \in \mathcal{A}(K)$, and ν a measure on K , we say that the triple (K, ν, Q) satisfies a weighted Bernstein-Markov property if for all $p_k \in \mathcal{P}_k$,

$$\|e^{-kQ} p_k\|_K \leq M_k \|e^{-kQ} p_k\|_{L^2(\nu)} \text{ with } \limsup_{k \rightarrow \infty} M_k^{1/k} = 1.$$

If K is locally regular and Q is continuous, for $\nu = (dd^c V_{K,Q})^n$ it is known that (K, ν, Q) satisfies a weighted Bernstein-Markov property [9]. For $Q = 0$ and K L -regular this was proved in [18].

For $\nu \in \mathcal{M}(K)$ and $Q \in \mathcal{A}(K)$, define

$$(3.1) \quad Z_k := Z_k(K, Q, \nu) := \int_K \cdots \int_K |VDM_k^Q(z_1, \dots, z_{N_k})|^2 d\nu(z_1) \cdots d\nu(z_{N_k}).$$

The following result is from [4] (see also [6]).

Proposition 3.1. *Let $K \subset \mathbb{C}^n$ be a compact set and let $Q \in \mathcal{A}(K)$. If ν is a measure on K with (K, ν, Q) satisfying a weighted Bernstein-Markov property, then*

$$(3.2) \quad \lim_{k \rightarrow \infty} Z_k^{\frac{1}{2kN_k}} = \bar{\delta}^Q(K).$$

Given ν as in Proposition 3.1, we define a probability measure $Prob_k$ on K^{N_k} via, for a Borel set $A \subset K^{N_k}$,

$$(3.3) \quad Prob_k(A) := \frac{1}{Z_k} \cdot \int_A |VDM_k^Q(z_1, \dots, z_{N_k})|^2 \cdot d\nu(z_1) \cdots d\nu(z_{N_k}).$$

We immediately obtain the following (cf., Proposition 3.3 of [12]).

Corollary 3.2. *Given $\eta > 0$, define*

$$(3.4) \quad A_{k,\eta} := \{(z_1, \dots, z_{N_k}) \in K^{N_k} : |VDM_k^Q(z_1, \dots, z_{N_k})|^2 \geq (\bar{\delta}^Q(K) - \eta)^{2kN_k}\}.$$

Then there exists $k^ = k^*(\eta)$ such that for all $k > k^*$,*

$$Prob_k(K^{N_k} \setminus A_{k,\eta}) \leq \left(1 - \frac{\eta}{2\bar{\delta}^Q(K)}\right)^{2kN_k}.$$

Proof. From Proposition 3.1, given $\epsilon > 0$,

$$Z_k \geq [\bar{\delta}^Q(K) - \epsilon]^{2kN_k}$$

for $k \geq k(\epsilon)$. Thus

$$\begin{aligned} \text{Prob}_k(K^{N_k} \setminus A_{k,\eta}) &= \\ \frac{1}{Z_k} \int_{K^{N_k} \setminus A_{k,\eta}} |VDM_k^Q(z_1, \dots, z_{N_k})|^2 d\nu(z_1) \cdots d\nu(z_{N_k}) \\ &\leq \frac{[\bar{\delta}^Q(K) - \eta]^{2kN_k}}{[\bar{\delta}^Q(K) - \epsilon]^{2kN_k}} \end{aligned}$$

if $k \geq k(\epsilon)$. Choosing $\epsilon < \eta/2$ and $k^* = k(\epsilon)$ gives the result. \square

Using (3.3), we get an induced probability measure \mathbf{P} on the infinite product space of arrays $\chi := \{X = \{x_j^{(k)}\}_{k=1,2,\dots; j=1,\dots,N_k} : x_j^{(k)} \in K\}$:

$$(\chi, \mathbf{P}) := \prod_{k=1}^{\infty} (K^{N_k}, \text{Prob}_k).$$

Corollary 3.3. *Let (K, ν, Q) satisfy a weighted Bernstein-Markov property. For \mathbf{P} -a.e. array $X = \{x_j^{(k)}\} \in \chi$,*

$$\nu_k := \frac{1}{N_k} \sum_{j=1}^{N_k} \delta_{x_j^{(k)}} \rightarrow \mu_{K,Q} \text{ weak-}^*.$$

Proof. (cf., [17], p. 211) From Theorem 2.4 it suffices to verify for \mathbf{P} -a.e.

$$(3.5) \quad \liminf_{k \rightarrow \infty} (|VDM_k^Q(x_1^{(k)}, \dots, x_{N_k}^{(k)})|)^{\frac{1}{kN_k}} = \bar{\delta}^Q(K).$$

Given $\eta > 0$, the condition that for a given array $X = \{x_j^{(k)}\}$ we have

$$\liminf_{k \rightarrow \infty} (|VDM_k^Q(x_1^{(k)}, \dots, x_{N_k}^{(k)})|)^{\frac{1}{kN_k}} \leq \bar{\delta}^Q(K) - \eta$$

means that $(x_1^{(k)}, \dots, x_{N_k}^{(k)}) \in K^{N_k} \setminus A_{k,\eta}$ for infinitely many k . Thus setting

$$E_k := \{X \in \chi : (x_1^{(k)}, \dots, x_{N_k}^{(k)}) \in K^{N_k} \setminus A_{k,\eta}\},$$

we have

$$\mathbf{P}(E_k) \leq \text{Prob}_k(K^{N_k} \setminus A_{k,\eta}) \leq (1 - \frac{\eta}{2\bar{\delta}^Q(K)})^{2kN_k}$$

and $\sum_{k=1}^{\infty} \mathbf{P}(E_k) < +\infty$. By the Borel-Cantelli lemma,

$$\mathbf{P}(\limsup_{k \rightarrow \infty} E_k) = 0.$$

Thus, with probability one, only finitely many E_k occur, and (3.5) follows. \square

A stronger version of Corollary 3.3 will be given in section 5. We now show that every compact set admits a measure satisfying a Bernstein-Markov property. Indeed, the following stronger statement is true.

Proposition 3.4. *Let $K \subset \mathbb{R}^n$. There exists a measure $\nu \in \mathcal{M}(K)$ such that for all complex-valued polynomials p of degree at most k in the (real) coordinates $x = (x_1, \dots, x_n)$ we have*

$$\|p\|_K \leq M_k \|p\|_{L^2(d\nu)}$$

where $\lim_{k \rightarrow \infty} M_k^{1/k} = 1$.

Proof. To construct ν , we first observe that if K is a finite set, any measure ν which puts positive mass at each point of K will work. If K has infinitely many points, for each $k = 1, 2, \dots$ let $m_k = \dim \mathcal{P}_k(K)$ where $\mathcal{P}_k(K)$ denotes the complex-valued polynomials on \mathbb{R}^n of degree at most k restricted to K . Then $\lim_{k \rightarrow \infty} m_k = \infty$ and $m_k \leq \binom{n+k}{k} = O(k^n)$. For each k , let

$$\nu_k := \frac{1}{m_k} \sum_{j=1}^{m_k} \delta(x_j^{(k)})$$

where $\{x_j^{(k)}\}_{j=1, \dots, m_k}$ is a set of Fekete points of order k for K ; i.e., if $\{e_1, \dots, e_{m_k}\}$ is any basis for $\mathcal{P}_k(K)$,

$$(3.6) \quad |\det[e_i(x_j^{(k)})]_{i,j=1, \dots, m_k}| = \max_{q_1, \dots, q_{m_k} \in K} |\det[e_i(q_j)]_{i,j=1, \dots, m_k}|.$$

Define

$$\nu := c \sum_{k=1}^{\infty} \frac{1}{k^2} \nu_k$$

where $c > 0$ is chosen so that $\nu \in \mathcal{M}(K)$. If $p \in \mathcal{P}_k(K)$, we have

$$p(x) = \sum_{j=1}^{m_k} p(x_j^{(k)}) l_j^{(k)}(x)$$

where, following the notation in (2.4) with N_k replaced by m_k but using real variables,

$$l_j^{(k)}(x) = \frac{VDM_k(x_1^{(k)}, \dots, x_{j-1}^{(k)}, x, x_{j+1}^{(k)}, \dots, x_{m_k}^{(k)})}{\det[e_i(x_j^{(k)})]} \in \mathcal{P}_k(K)$$

so $l_j^{(k)}(x_j^{(k)}) = \delta_{jk}$. Since $\|l_j^{(k)}\|_K = 1$ from (3.6) we have

$$\|p\|_K \leq \sum_{j=1}^{m_k} |p(x_j^{(k)})|.$$

On the other hand,

$$\begin{aligned} \|p\|_{L^2(d\nu)} &\geq \|p\|_{L^1(d\nu)} \geq \frac{c}{k^2} \int_K |p| d\nu_k \\ &= \frac{c}{m_k k^2} \sum_{j=1}^{m_k} |p(x_j^{(k)})|. \end{aligned}$$

Thus we have

$$\|p\|_K \leq \frac{m_k k^2}{c} \|p\|_{L^2(d\nu)}.$$

□

Since the holomorphic polynomials in \mathbb{C}^n are a subset of the complex-valued polynomials in the underlying real coordinates in $\mathbb{C}^n = \mathbb{R}^{2n}$, we immediately obtain the following.

Corollary 3.5. *Let $K \subset \mathbb{C}^n$ be a compact set. Then there exists a measure $\nu \in \mathcal{M}(K)$ such that (K, ν) satisfies a Bernstein-Markov property.*

Remark 3.6. Note that the measure ν constructed in Proposition 3.4 (and hence in Corollary 3.5) is discrete. In [7] it was shown using a different procedure that any L -regular compact set admits a discrete measure satisfying a Bernstein-Markov property.

Definition 3.7. Given a compact set $K \subset \mathbb{C}^n$, we say $\nu \in \mathcal{M}(K)$ satisfies a *strong Bernstein-Markov property* if (K, ν, Q) satisfies a weighted Bernstein-Markov property for all $Q \in C(K)$.

Corollary 3.8. *Let $K \subset \mathbb{C}^n$ be a nonpluripolar compact set. Then there exists a measure $\nu \in \mathcal{M}(K)$ satisfying a strong Bernstein-Markov property.*

Proof. We consider $\mathbb{C}^n = \mathbb{R}^{2n} \subset \mathbb{C}^{2n}$ and we use Corollary 3.5 to construct a measure ν on K such that (K, ν) satisfies a Bernstein-Markov property with respect to holomorphic polynomials on \mathbb{C}^{2n} . Theorem 3.2 of [9] then shows that (K, ν, Q) satisfies a weighted Bernstein-Markov property for all $Q \in C(K)$. \square

It is convenient to have a simple sufficient condition for a measure to satisfy the strong Bernstein-Markov property. We say that (K, ν) satisfies a *mass-density property* if there exists $T > 0$ so that

$$\nu(B(z_0, r)) \geq r^T$$

for all $z_0 \in K$ and all $r < r(z_0)$ where $r(z_0) > 0$. In [11] it was shown that for K L -regular, this property (indeed, a weaker mass-density property will suffice) implies that (K, ν) satisfies a Bernstein-Markov property. Thus if $K \subset \mathbb{R}^n \subset \mathbb{C}^n$ is L -regular and (K, ν) satisfies a mass-density property, then (K, ν) satisfies a strong Bernstein-Markov property. In particular, if $K = \overline{D}$ when D is a bounded domain in \mathbb{R}^n with C^1 -boundary, any ν which is a positive, continuous multiple of Lebesgue measure on D is a strong Bernstein-Markov measure for K .

4. Relation between E^* and J, J^Q, W, W^Q functionals.

We begin by sharpening Theorem 2.2 in [13] in two ways:

- (1) the result applies to all *nonpluripolar compact sets* $K \subset \mathbb{C}^n$;
- (2) the functionals defined using a “lim sup” and a “lim inf” coincide (see Definitions 4.1 and 4.2) – and this is the essence of the large deviation principle in Theorem 5.1.

We remark that $\mathcal{M}(K)$, with the weak-* topology, is a Polish space; i.e., a separable, complete metrizable space. A neighborhood basis of $\mu \in \mathcal{M}(K)$ is given by sets of the form

$$(4.1) \quad G(\mu, k, \epsilon) := \left\{ \sigma \in \mathcal{M}(K) : \left| \int_K (\operatorname{Re} z)^\alpha (\operatorname{Im} z)^\beta (d\mu - d\sigma) \right| < \epsilon \right.$$

for $0 \leq |\alpha| + |\beta| \leq k$ where $\operatorname{Re} z = (\operatorname{Re} z_1, \dots, \operatorname{Re} z_n)$.

Fix a nonpluripolar compact set K and a strong Bernstein-Markov measure ν . Given $G \subset \mathcal{M}(K)$, for each $s = 1, 2, \dots$ we set

$$(4.2) \quad \tilde{G}_s := \left\{ \mathbf{a} = (a_1, \dots, a_s) \in K^s : \frac{1}{s} \sum_{j=1}^s \delta_{a_j} \in G \right\}.$$

Then we define, for $k = 1, 2, \dots$,

$$J_k(G) := \left[\int_{\tilde{G}_{N_k}} |VDM_k(\mathbf{a})|^2 d\nu(\mathbf{a}) \right]^{1/2kN_k}$$

and

$$W_k(G) := \sup\{|VDM_k(\mathbf{a})|^{1/kN_k} : \mathbf{a} \in \tilde{G}_{N_k}\}.$$

Definition 4.1. For $\mu \in \mathcal{M}(K)$ we define

$$\begin{aligned} \overline{J}(\mu) &:= \inf_{G \ni \mu} \overline{J}(G) \text{ where } \overline{J}(G) := \limsup_{k \rightarrow \infty} J_k(G); \\ \underline{J}(\mu) &:= \inf_{G \ni \mu} \underline{J}(G) \text{ where } \underline{J}(G) := \liminf_{k \rightarrow \infty} J_k(G); \end{aligned}$$

and

$$\begin{aligned} \overline{W}(\mu) &:= \inf_{G \ni \mu} \overline{W}(G) \text{ where } \overline{W}(G) := \limsup_{k \rightarrow \infty} W_k(G); \\ \underline{W}(\mu) &:= \inf_{G \ni \mu} \underline{W}(G) \text{ where } \underline{W}(G) := \liminf_{k \rightarrow \infty} W_k(G). \end{aligned}$$

Here the infima are taken over all neighborhoods G of the measure μ in $\mathcal{M}(K)$. Note that $\overline{W}, \underline{W}$ are independent of ν but, a priori, $\overline{J}, \underline{J}$ depend on ν . These functionals are clearly nonnegative but can take the value zero. The weighted versions of these functionals are defined for $Q \in \mathcal{A}(K)$ starting with

$$(4.3) \quad J_k^Q(G) := \left[\int_{\tilde{G}_{N_k}} |VDM_k^Q(\mathbf{a})|^2 d\nu(\mathbf{a}) \right]^{1/2kN_k}$$

and

$$W_k^Q(G) := \sup\{|VDM_k^Q(\mathbf{a})|^{1/kN_k} : \mathbf{a} \in \tilde{G}_{N_k}\}.$$

Definition 4.2. For $\mu \in \mathcal{M}(K)$ we define

$$\begin{aligned} \overline{J}^Q(\mu) &:= \inf_{G \ni \mu} \overline{J}^Q(G) \text{ where } \overline{J}^Q(G) := \limsup_{k \rightarrow \infty} J_k^Q(G); \\ \underline{J}^Q(\mu) &:= \inf_{G \ni \mu} \underline{J}^Q(G) \text{ where } \underline{J}^Q(G) := \liminf_{k \rightarrow \infty} J_k^Q(G); \end{aligned}$$

and

$$\begin{aligned} \overline{W}^Q(\mu) &:= \inf_{G \ni \mu} \overline{W}^Q(G) \text{ where } \overline{W}^Q(G) := \limsup_{k \rightarrow \infty} W_k^Q(G); \\ \underline{W}^Q(\mu) &:= \inf_{G \ni \mu} \underline{W}^Q(G) \text{ where } \underline{W}^Q(G) := \liminf_{k \rightarrow \infty} W_k^Q(G). \end{aligned}$$

It is straightforward from the definitions (cf., [13]) that

$$(1) \quad \overline{J}^Q(\mu) \leq \overline{W}^Q(\mu) \leq \delta^Q(K) \text{ for } Q \in \mathcal{A}(K);$$

- (2) $\bar{J}(\mu) = \bar{J}^Q(\mu) \cdot e^{\int_K Q d\mu}$ and $\bar{W}(\mu) = \bar{W}^Q(\mu) \cdot e^{\int_K Q d\mu}$ for $Q \in C(K)$; hence
- (3) $\log \bar{J}(\mu) \leq \log \bar{W}(\mu) \leq \inf_{v \in C(K)} [\log \bar{\delta}^v(K) + \int_K v d\mu]$; similarly
- (4) $\log \bar{J}^Q(\mu) \leq \log \bar{W}^Q(\mu) \leq \inf_{v \in C(K)} [\log \bar{\delta}^v(K) + \int_K v d\mu] - \int_K Q d\mu$ for $Q \in C(K)$.

Properties (1)-(4) hold for the functionals $\underline{J}, \underline{W}, \underline{J}^Q, \underline{W}^Q$ using the same proofs as in section 3 of [13]. The uppersemicontinuity of all eight functionals on $\mathcal{M}(K)$ (with the weak-* topology) follows as well. We note the following.

Proposition 4.3. *The measure $\mu_{K,Q}$ is the unique maximizer of the functional $\mu \rightarrow \bar{W}^Q(\mu)$ and $\bar{W}^Q(\mu_{K,Q}) = \bar{\delta}^Q(K)$.*

Proof. It follows from its definition that

$$\bar{W}^Q(\mu) = \sup \left\{ \limsup_{k \rightarrow \infty} |VDM_k^Q(\mathbf{a}^{(k)})|^{1/kN_k} \right\} \leq \bar{\delta}^Q(K)$$

where the supremum is taken over all arrays $\{\mathbf{a}^{(k)}\}_{k=1,2,\dots}$ of N_k -tuples $\mathbf{a}^{(k)}$ whose normalized counting measures μ_k converges to μ weak*. If $\bar{W}^Q(\mu) = \bar{\delta}^Q(K)$ then there is an asymptotic weighted Fekete array $\{\mathbf{a}^{(k)}\}_{k=1,2,\dots}$ as in the statement of Theorem 2.4 with $\mu_k \rightarrow \mu$ weak*. From this theorem, $\mu = \mu_{K,Q}$. \square

Theorems 4.4 and 4.6 show that the inequalities in (3) and (4) are equalities, and that the $\bar{J}, \bar{W}, \bar{J}^Q, \bar{W}^Q$ functionals coincide with their $\underline{J}, \underline{W}, \underline{J}^Q, \underline{W}^Q$ counterparts.

Theorem 4.4. *Let $K \subset \mathbb{C}^n$ be a nonpluripolar compact set and $Q \in C(K)$. Then for any $\mu \in \mathcal{M}(K)$,*

$$(4.4) \quad \log \bar{W}(\mu) = \log \underline{W}(\mu) = \inf_{v \in C(K)} [\log \bar{\delta}^v(K) + \int_K v d\mu] \text{ and}$$

$$(4.5) \quad \log \bar{W}^Q(\mu) = \log \underline{W}^Q(\mu) = \inf_{v \in C(K)} [\log \bar{\delta}^v(K) + \int_K v d\mu] - \int_K Q d\mu.$$

Remark 4.5. Note that using (2.7) and (2.6) equation (4.4) says that

$$-\log \bar{W}(\mu) = -\log \underline{W}(\mu) = E^*(\mu) - \int_K u_0 d\mu - E(V_T)$$

while (4.5) says that

$$-\log \overline{W}^Q(\mu) = -\log \underline{W}^Q(\mu) = E^*(\mu) + \int_K (Q - u_0) d\mu - E(V_T).$$

Thus from (2.8) we recover Proposition 4.3.

Proof. It suffices to prove (4.4) as then (4.5) follows from property (2). We have the upper bound

$$(4.6) \quad \log \underline{W}(\mu) \leq \log \overline{W}(\mu) \leq \inf_{v \in C(K)} [\log \overline{\delta}^v(K) + \int_K v d\mu]$$

from (3).

For the lower bound, we consider cases.

Case I: $\mu = \mu_{K,v}$ for some $v \in C(K)$.

Using Theorem 2.4, if we consider arrays of points $\{x_1^{(k)}, \dots, x_{N_k}^{(k)}\}_{k=1,2,\dots}$ in K for which

$$\lim_{k \rightarrow \infty} (|VDM_k^v(x_1^{(k)}, \dots, x_{N_k}^{(k)})|)^{\frac{1}{kN_k}} = \overline{\delta}^v(K),$$

we have $\frac{1}{N_k} \sum_{j=1}^{N_k} \delta_{x_j^{(k)}} \rightarrow \mu_{K,v}$ weak-*. Thus for any neighborhood G of $\mu_{K,v}$ we have $\overline{\delta}^v(K) \leq \underline{W}^v(\mu_{K,v}, G)$; hence

$$(4.7) \quad \underline{W}^v(\mu_{K,v}) = \overline{W}^v(\mu_{K,v}) = \overline{\delta}^v(K).$$

Applying (2) to (4.7) we obtain (4.4) for $\mu = \mu_{K,v}$:

$$(4.8) \quad \log \underline{W}(\mu_{K,v}) = \log \overline{W}(\mu_{K,v}) = \log \overline{\delta}^v(K) + \int_K v d\mu_{K,v}.$$

Case II: $\mu \in \mathcal{M}(K)$ with the property that $E^*(\mu) < \infty$.

From Theorem 2.3 there exists $u \in L(\mathbb{C}^n)$ with $\mu = (dd^c u)^n$ and $\int_K u d\mu > -\infty$ (see also Proposition 5.6 of [13]). However, since u is only usc on K , μ is not necessarily of the form $\mu_{K,v}$ for some $v \in C(K)$. Following the argument in Proposition 4.3 of [13], taking a sequence of functions $\{Q_j\} \subset C(K)$ with $Q_j \downarrow u$ on K , the weighted extremal functions V_{K,Q_j}^* decrease to u on \mathbb{C}^n ;

$$\mu_j := (dd^c V_{K,Q_j})^n \rightarrow \mu = (dd^c u)^n \text{ weak-*};$$

and

$$(4.9) \quad \lim_{j \rightarrow \infty} \int_K V_{K,Q_j}^* d\mu_j = \lim_{j \rightarrow \infty} \int_K V_{K,Q_j}^* d\mu = \int_K u d\mu.$$

(Note that in this case, since V_{K,Q_j} is not necessarily continuous, we get $\tilde{u} := \lim_{j \rightarrow \infty} V_{K,Q_j}^* \geq u$ on \mathbb{C}^n and $\tilde{u} = u$ q.e. on K which suffices for the application of the domination principle, Corollary A.2 of [13]). From the previous case (4.8), we have

$$\log \overline{W}(\mu_j) = \log \underline{W}(\mu_j) = \log \overline{\delta}^{Q_j}(K) + \int_K Q_j d\mu_j.$$

Using uppersemicontinuity of the functional $\mu \rightarrow \underline{W}(\mu)$,

$$\limsup_{j \rightarrow \infty} \underline{W}(\mu_j) = \limsup_{j \rightarrow \infty} \overline{W}(\mu_j) \leq \underline{W}(\mu).$$

Since $Q_j \downarrow u$ on K ,

$$(4.10) \quad \limsup_{j \rightarrow \infty} \log \overline{\delta}^{Q_j}(K) = \lim_{j \rightarrow \infty} \log \overline{\delta}^{Q_j}(K)$$

exists. Since $V_{K,Q_j}^* = Q_j$ q.e. on $\text{supp} \mu_j$, from (4.9),

$$\lim_{j \rightarrow \infty} \int_K Q_j d\mu_j = \int_K u d\mu.$$

Thus, together with (4.10), we see that

$$\lim_{j \rightarrow \infty} \log \underline{W}(\mu_j) = \lim_{j \rightarrow \infty} (\log \overline{\delta}^{Q_j}(K) + \int_K Q_j d\mu_j) := M$$

exists and is less than or equal to $\log \underline{W}(\mu)$. We want to show that

$$(4.11) \quad \inf_v [\log \overline{\delta}^v(K) + \int_K v d\mu] \leq M.$$

By monotone convergence, $\lim_{j \rightarrow \infty} \int_K Q_j d\mu = \int_K u d\mu$. Since we also have $\lim_{j \rightarrow \infty} \int_K Q_j d\mu_j = \int_K u d\mu$, given $\epsilon > 0$, for $j \geq j_0(\epsilon)$,

$$\int_K Q_j d\mu_j \geq \int_K Q_j d\mu - \epsilon \text{ and } \log \underline{W}(\mu_j) < M + \epsilon.$$

Hence for such j ,

$$\begin{aligned} \inf_w [\log \overline{\delta}^w(K) + \int_K w d\mu] &\leq \log \overline{\delta}^{Q_j}(K) + \int_K Q_j d\mu \\ &\leq \log \overline{\delta}^{Q_j}(K) + \int_K Q_j d\mu_j + \epsilon = \log \underline{W}(\mu_j) + \epsilon < M + 2\epsilon, \end{aligned}$$

yielding (4.11). This finishes the proof in Case II. Note that from (2.7) we have

$$\inf_v [\log \overline{\delta}^v(K) + \int_K v d\mu] > -\infty$$

in this case.

Case III: $\mu \in \mathcal{M}(K)$ with the property that $\inf_v [\log \bar{\delta}^v(K) + \int_K v d\mu] = -\infty$.

As in [13], we must show that $\bar{W}(\mu) = 0$. This follows trivially from the upper bound (4.6). \square

We next turn to the $\bar{J}, \underline{J}, \bar{J}^Q$ and \underline{J}^Q functionals. The key step in the proof of Theorem 4.6 is to verify (4.7) for $\bar{J}^v(\mu_{K,v})$ and $\underline{J}^v(\mu_{K,v})$.

Theorem 4.6. *Let $K \subset \mathbb{C}^n$ be a nonpluripolar compact set and let ν satisfy a strong Bernstein-Markov property. Fix $Q \in C(K)$. Then for any $\mu \in \mathcal{M}(K)$,*

$$(4.12) \quad \begin{aligned} \log \bar{J}(\mu) &= \log \bar{W}(\mu) = \log \underline{J}(\mu) = \log \underline{W}(\mu) \\ &= \inf_{v \in C(K)} [\log \bar{\delta}^v(K) + \int_K v d\mu] \end{aligned}$$

and

$$(4.13) \quad \begin{aligned} \log \bar{J}^Q(\mu) &= \log \bar{W}^Q(\mu) = \log \underline{J}^Q(\mu) = \log \underline{W}^Q(\mu) \\ &= \inf_{v \in C(K)} [\log \bar{\delta}^v(K) + \int_K v d\mu] - \int_K Q d\mu. \end{aligned}$$

Proof. As in the previous proof, it suffices to prove (4.12) since (4.13) follows from property (2). Again we have the upper bound

$$\log \bar{J}(\mu) \leq \log \bar{W}(\mu) \leq \inf_{v \in C(K)} [\log \bar{\delta}^v(K) + \int_K v d\mu]$$

from (3); for the lower bound, we need only consider Case I where $\mu = \mu_{K,v}$ for $v \in C(K)$. We show the analogue of (4.8) for \bar{J}, \underline{J} :

$$(4.14) \quad \log \bar{J}(\mu_{K,v}) = \log \underline{J}(\mu_{K,v}) = \log \bar{\delta}^v(K) + \int_K v d\mu_{K,v}.$$

This shows, in particular, that

$$\log \bar{J}(\mu_{K,v}) = \log \bar{W}(\mu_{K,v}) = \log \underline{J}(\mu_{K,v}) = \log \underline{W}(\mu_{K,v})$$

and hence we have

$$\log \bar{J}(\mu) = \log \bar{W}(\mu) = \log \underline{J}(\mu) = \log \underline{W}(\mu)$$

for arbitrary $\mu \in \mathcal{M}(K)$ following the proof of Theorem 4.4. This proves (4.12). To prove (4.14), we first verify the following.

Claim: Fix a neighborhood G of $\mu_{K,v}$. We can find k_0 such that

$$(4.15) \quad A_{k,1/k} \subset \tilde{G}_{N_k} \text{ for all } k \geq k_0$$

where $A_{k,1/k}$ is defined in (3.4) with $Q = v$ and $\eta = 1/k$.

We prove (4.15) by contradiction: if false, then there is a sequence $\{k_j\}$ with $k_j \uparrow \infty$ such that for all j sufficiently large we can find a point $x^j = (x_1^j, \dots, x_{N_{k_j}}^j) \in A_{k_j,1/k_j} \setminus \tilde{G}_{N_{k_j}}$. But $\mu_j := \frac{1}{N_{k_j}} \sum_{i=1}^{N_{k_j}} \delta_{x_i^j} \notin \tilde{G}_{N_{k_j}}$ for j sufficiently large contradicts Theorem 2.4 since $\mu_j \in A_{k_j,1/k_j}$ and $k_j \uparrow \infty$ imply $\mu_j \rightarrow \mu_{K,v}$ weak-*. This proves the claim.

We observe that since $N_k = \binom{n+k}{k}$

$$\left(1 - \frac{1/k}{2\bar{\delta}^v(K)}\right)^{2kN_k} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Using (4.15) and Corollary 3.2,

$$\begin{aligned} & \frac{1}{Z_k} \int_{\tilde{G}_{N_k}} |VDM_k^v(z_1, \dots, z_{N_k})|^2 \cdot d\nu(z_1) \cdots d\nu(z_{N_k}) \\ & \geq \frac{1}{Z_k} \int_{A_{k,1/k}} |VDM_k^v(z_1, \dots, z_{N_k})|^2 \cdot d\nu(z_1) \cdots d\nu(z_{N_k}) \\ & \geq 1 - \left(1 - \frac{1/k}{2\bar{\delta}^v(K)}\right)^{2kN_k} \rightarrow 1 \text{ as } k \rightarrow \infty. \end{aligned}$$

Since ν satisfies a strong Bernstein-Markov property and $v \in C(K)$, using Proposition 3.1 we conclude that

$$\begin{aligned} \liminf_{k \rightarrow \infty} \frac{1}{2kN_k} \log \int_{\tilde{G}_{N_k}} |VDM_k^v(z_1, \dots, z_{N_k})|^2 d\nu(z_1) \cdots d\nu(z_{N_k}) \\ \geq \log \bar{\delta}^v(K). \end{aligned}$$

Taking the infimum over all neighborhoods G of $\mu_{K,v}$ we obtain

$$\log \underline{J}^v(\mu_{K,v}) \geq \log \bar{\delta}^v(K).$$

Thus we have the version of (4.7) with \bar{J}^v and \underline{J}^v :

$$(4.16) \quad \log \underline{J}^v(\mu_{K,v}) = \log \bar{J}^v(\mu_{K,v}) = \log \bar{\delta}^v(K).$$

Using (2) with $\mu = \mu_{K,v}$ we obtain (4.14). □

Remark 4.7. From now on, we simply use the notation J, J^Q, W, W^Q without the overline or underline.

5. Large deviation.

In this section we take $K \subset \mathbb{C}^n$ a nonpluripolar compact set and a measure ν satisfying a strong Bernstein-Markov property. Fix $Q \in C(K)$. For $x_1, \dots, x_{N_k} \in K$, we get a discrete probability measure $\kappa_k(x) := \frac{1}{N_k} \sum_{j=1}^{N_k} \delta_{x_j}$. Define $j_k : K^{N_k} \rightarrow \mathcal{M}(K)$ via

$$j_k(x_1, \dots, x_{N_k}) = \kappa_k(x) := \frac{1}{N_k} \sum_{j=1}^{N_k} \delta_{x_j}.$$

From (3.3), $\sigma_k := (j_k)_*(\text{Prob}_k)$ is a probability measure on $\mathcal{M}(K)$: for a Borel set $B \subset \mathcal{M}(K)$,

$$(5.1) \quad \sigma_k(B) = \frac{1}{Z_k} \int_{\tilde{B}_{N_k}} |VDM_k^Q(x_1, \dots, x_{N_k})|^2 d\nu(x_1) \cdots d\nu(x_{N_k})$$

(recall (3.3) and (4.2); here, $Z_k := Z_k(K, Q, \nu)$). In addition, suppose we have a function $F : \mathbb{R} \rightarrow \mathbb{R}$ and a function $v \in C(K)$. We write, for $\mu \in \mathcal{M}(K)$,

$$\langle v, \mu \rangle := \int_K v d\mu$$

and then

$$(5.2) \quad \int_{\mathcal{M}(K)} F(\langle v, \mu \rangle) d\sigma_k(\mu) :=$$

$$\frac{1}{Z_k} \int_K \cdots \int_K |VDM_k^Q(x_1, \dots, x_{N_k})|^2 F\left(\frac{1}{N_k} \sum_{j=1}^{N_k} v(x_j)\right) d\nu(x_1) \cdots d\nu(x_{N_k}).$$

Theorem 5.1. *The sequence $\{\sigma_k = (j_k)_*(\text{Prob}_k)\}$ of probability measures on $\mathcal{M}(K)$ satisfies a **large deviation principle** with speed $2kN_k$ and good rate function $\mathcal{I} := \mathcal{I}_{K,Q}$ where, for $\mu \in \mathcal{M}(K)$,*

$$(5.3) \quad \mathcal{I}(\mu) := \log J^Q(\mu_{K,Q}) - \log J^Q(\mu) = \log W^Q(\mu_{K,Q}) - \log W^Q(\mu).$$

This means that $\mathcal{I} : \mathcal{M}(K) \rightarrow [0, \infty]$ is a lowersemicontinuous mapping such that the sublevel sets $\{\mu \in \mathcal{M}(K) : \mathcal{I}(\mu) \leq \alpha\}$ are compact in the weak-* topology on $\mathcal{M}(K)$ for all $\alpha \geq 0$ (\mathcal{I} is “good”) satisfying (5.4) and (5.5):

Definition 5.2. The sequence $\{\mu_k\}$ of probability measures on $\mathcal{M}(K)$ satisfies a **large deviation principle** (LDP) with good rate function \mathcal{I} and speed $2kN_k$ if for all measurable sets $\Gamma \subset \mathcal{M}(K)$,

$$(5.4) \quad -\inf_{\mu \in \Gamma^0} \mathcal{I}(\mu) \leq \liminf_{k \rightarrow \infty} \frac{1}{2kN_k} \log \mu_k(\Gamma) \text{ and}$$

$$(5.5) \quad \limsup_{k \rightarrow \infty} \frac{1}{2kN_k} \log \mu_k(\Gamma) \leq -\inf_{\mu \in \bar{\Gamma}} \mathcal{I}(\mu).$$

In the setting of $\mathcal{M}(K)$, to prove a LDP it suffices to work with a base for the weak-* topology. The following is a special case of a basic general existence result for a LDP given in Theorem 4.1.11 in [14].

Proposition 5.3. *Let $\{\sigma_\epsilon\}$ be a family of probability measures on $\mathcal{M}(K)$. Let \mathcal{B} be a base for the topology of $\mathcal{M}(K)$. For $\mu \in \mathcal{M}(K)$ let*

$$\mathcal{I}(\mu) := -\inf_{\{G \in \mathcal{B}; \mu \in G\}} \left(\liminf_{\epsilon \rightarrow 0} \epsilon \log \sigma_\epsilon(G) \right).$$

Suppose for all $\mu \in \mathcal{M}(K)$,

$$\mathcal{I}(\mu) := -\inf_{\{G \in \mathcal{B}; \mu \in G\}} \left(\limsup_{\epsilon \rightarrow 0} \epsilon \log \sigma_\epsilon(G) \right).$$

Then $\{\sigma_\epsilon\}$ satisfies a LDP with rate function $\mathcal{I}(\mu)$ and speed $1/\epsilon$.

We give our first proof of Theorem 5.1 using Theorem 4.6.

Proof. As a base \mathcal{B} for the topology of $\mathcal{M}(K)$, we can take the sets from (4.1) or simply all open sets. For $\{\sigma_\epsilon\}$, we take the sequence of probability measures $\{\sigma_k\}$ on $\mathcal{M}(K)$ and we take $\epsilon = \frac{1}{2kN_k}$. For $G \in \mathcal{B}$,

$$\frac{1}{2kN_k} \log \sigma_k(G) = \log J_k^Q(G) - \frac{1}{2kN_k} \log Z_k$$

using (4.3) and (5.1). From Proposition 3.1, and (4.16) with $v = Q$,

$$\lim_{k \rightarrow \infty} \frac{1}{2kN_k} \log Z_k = \log \bar{\delta}^Q(K) = \log J^Q(\mu_{K,Q});$$

and by Theorem 4.6, if $\inf_v [\log \bar{\delta}^v(K) + \int_K v d\mu] > -\infty$,

$$\inf_{G \ni \mu} \limsup_{k \rightarrow \infty} \log J_k^Q(G) = \inf_{G \ni \mu} \liminf_{k \rightarrow \infty} \log J_k^Q(G) = \log J^Q(\mu).$$

If, on the other hand, $\inf_v [\log \bar{\delta}^v(K) + \int_K v d\mu] = -\infty$, then

$$\lim_{k \rightarrow \infty} \log J_k^Q(G) = -\infty.$$

Thus by Proposition 5.3 and Theorem 4.6, $\{\sigma_k\}$ satisfies an LDP with rate function

$$\mathcal{I}(\mu) := \log J^Q(\mu_{K,Q}) - \log J^Q(\mu) = \log W^Q(\mu_{K,Q}) - \log W^Q(\mu)$$

and speed $2kN_k$. This rate function is good since $\mathcal{M}(K)$ is compact. \square

Proposition 4.3 shows that $\mu_{K,Q}$ maximizes the functional

$$(5.6) \quad \mu \rightarrow \log W^Q(\mu)$$

over all $\mu \in \mathcal{M}(K)$. Thus

$$(5.7) \quad \mathcal{I}_{K,Q}(\mu) \geq 0 \text{ with } \mathcal{I}_{K,Q}(\mu) = 0 \iff \mu = \mu_{K,Q}.$$

To summarize, $\mathcal{I}_{K,Q}$ is a good rate function with unique minimizer $\mu_{K,Q}$.

There is a converse to Proposition 5.3, Theorem 4.1.18 in [14]. For $\mathcal{M}(K)$, it reads as follows:

Proposition 5.4. *Let $\{\sigma_\epsilon\}$ be a family of probability measures on $\mathcal{M}(K)$. Suppose that $\{\sigma_\epsilon\}$ satisfies a LDP with rate function $\mathcal{I}(\mu)$ and speed $1/\epsilon$. Then for any base \mathcal{B} for the topology of $\mathcal{M}(K)$ and any $\mu \in \mathcal{M}(K)$*

$$\begin{aligned} \mathcal{I}(\mu) &:= - \inf_{\{G \in \mathcal{B}: \mu \in G\}} \left(\liminf_{\epsilon \rightarrow 0} \epsilon \log \sigma_\epsilon(G) \right) \\ &= - \inf_{\{G \in \mathcal{B}: \mu \in G\}} \left(\limsup_{\epsilon \rightarrow 0} \epsilon \log \sigma_\epsilon(G) \right). \end{aligned}$$

Remark 5.5. This shows that, starting with a strong Bernstein-Markov measure ν and the corresponding sequence of probability measures $\{\sigma_k\}$ on $\mathcal{M}(K)$ in (5.1), the existence of an LDP with rate function $\mathcal{I}(\mu)$ and speed $2kN_k$ implies that necessarily

$$(5.8) \quad \mathcal{I}(\mu) = \log J^Q(\mu_{K,Q}) - \log J^Q(\mu).$$

We mention that uniqueness of the rate function is basic (cf., Lemma 4.1.4 of [14]). The main ingredient utilized here is Theorem 2.4 on the distribution of asymptotic weighted Fekete points. This in turn hinges on the differentiability result Theorem 2.1. On the other hand, in potential-theoretic situations where the analogue of Theorem 2.4 is easily obtained, this approach to a LDP is fairly straightforward. We illustrate this in the univariate case in Section 6.

We turn to the second proof of Theorem 5.1. This follows from Corollary 4.6.14 in [14], which is a general version of the Gärtner-Ellis theorem. This approach was brought to our attention by S. Boucksom. Again, we state the version of the [14] result for $\mathcal{M}(K)$.

Proposition 5.6. *Let $\{\sigma_\epsilon\}$ be a family of Borel probability measures on $C(K)^*$ equipped with the weak- $*$ topology. Suppose for each $\lambda \in C(K)$, the limit*

$$\Lambda(\lambda) := \lim_{\epsilon \rightarrow 0} \epsilon \log \int_{C(K)^*} e^{\lambda(x)/\epsilon} d\sigma_\epsilon(x)$$

exists as a finite real number and assume Λ is Gateaux differentiable; i.e., for each $\lambda, \theta \in C(K)$, the function $f(t) := \Lambda(\lambda + t\theta)$ is differentiable at $t = 0$. Then $\{\sigma_\epsilon\}$ satisfies an LDP in $C(K)^$ with the convex, good rate function Λ^* .*

Here

$$\Lambda^*(x) := \sup_{\lambda \in C(K)} (\langle \lambda, x \rangle - \Lambda(\lambda)),$$

is the Legendre transform of Λ . The upper bound (5.5) in the LDP holds with rate function Λ^* under the assumption that the limit $\Lambda(\lambda)$ exists and is finite; the Gateaux differentiability of Λ is needed for the lower bound (5.4). We proceed with the second proof of Theorem 5.1.

Proof. We show that for each $v \in C(K)$,

$$\lim_{k \rightarrow \infty} \frac{1}{2kN_k} \log \int_{C(K)^*} e^{2kN_k \langle v, \mu \rangle} d\sigma_k(\mu)$$

exists as a finite real number. First, since σ_k is in $\mathcal{M}(K)$, the integral can be taken over $\mathcal{M}(K)$. Consider

$$\frac{1}{2kN_k} \log \int_{\mathcal{M}(K)} e^{2kN_k \langle v, \mu \rangle} d\sigma_k(\mu).$$

By (5.2), this is equal to

$$\frac{1}{2kN_k} \log \frac{1}{Z_k} \cdot \int_{K^{N_k}} |VDM_k^{Q-v}(x_1, \dots, x_{N_k})|^2 d\nu(x_1) \cdots d\nu(x_{N_k}).$$

Recall that

$$Z_k = \int_{K^{N_k}} |VDM_k^Q(x_1, \dots, x_{N_k})|^2 d\nu(x_1) \cdots d\nu(x_{N_k}).$$

Define

$$\tilde{Z}_k := \int_{K^{N_k}} |VDM_k^{Q-v}(x_1, \dots, x_{N_k})|^2 d\nu(x_1) \cdots d\nu(x_{N_k}).$$

Then we have

$$\lim_{k \rightarrow \infty} \tilde{Z}_k^{\frac{1}{2kN_k}} = \bar{\delta}^{Q-v}(K) \text{ and } \lim_{k \rightarrow \infty} Z_k^{\frac{1}{2kN_k}} = \bar{\delta}^Q(K)$$

from (3.2) in Proposition 3.1 and the assumption that (K, ν, \tilde{Q}) satisfies the weighted Bernstein-Markov property for *all* $\tilde{Q} \in C(K)$. Thus

$$(5.9) \quad \Lambda(v) = \lim_{k \rightarrow \infty} \frac{1}{2kN_k} \log \frac{\tilde{Z}_k}{Z_k} = \log \frac{\bar{\delta}^{Q-v}(K)}{\bar{\delta}^Q(K)}.$$

Define now, for $v, v' \in C(K)$,

$$f(t) := E(V_{K, Q-(v+tv')}).$$

Theorem 2.1 and (2.6) give that Λ is Gateaux differentiable and Proposition 5.6 gives that Λ^* is a rate function on $C(K)^*$.

Since each σ_k has support in $\mathcal{M}(K)$, it follows from (5.4) and (5.5) in Definition 5.2 of an LDP with $\Gamma \subset C(K)^*$ that for $\mu \in C(K)^* \setminus \mathcal{M}(K)$, $\Lambda^*(\mu) = +\infty$. By Lemma 4.1.5 (b) of [14], the restriction of Λ^* to $\mathcal{M}(K)$ is a rate function. Since $\mathcal{M}(K)$ is compact, it is a good rate function. Being a Legendre transform, Λ^* is convex.

To compute Λ^* and demonstrate (5.3), we have, using (5.9) and (2.6),

$$\Lambda^*(\mu) = \sup_{v \in C(K)} \left(\int_K v d\mu - E(V_{K, Q}) + E(V_{K, Q-v}) \right).$$

Thus

$$\begin{aligned} \Lambda^*(\mu) + E(V_{K, Q}) &= \sup_{v \in C(K)} \left(\int_K v d\mu + E(V_{K, Q-v}) \right) \\ &= \sup_{u \in C(K)} \left(E(V_{K, Q+u}) - \int_K u d\mu \right) \text{ (taking } u = -v). \end{aligned}$$

Rearranging and replacing u in the supremum by $v = u + Q$,

$$\begin{aligned} \Lambda^*(\mu) &= \sup_{u \in C(K)} \left(E(V_{K, Q+u}) - \int_K u d\mu \right) - E(V_{K, Q}) \\ (5.10) \quad &= \sup_{v \in C(K)} \left(E(V_{K, v}) - \int_K v d\mu \right) - \left[E(V_{K, Q}) - \int_K Q d\mu \right]. \end{aligned}$$

Using (2.6) and Theorem 4.6, we obtain (5.3). □

Remark 5.7. From the definition of E^* in Definition 2.2, equation (5.10) gives, after some manipulation,

$$(5.11) \quad \Lambda^*(\mu) = E^*(\mu) + \int_K (Q - u_0) d\mu - E(V_{K, Q}).$$

Recall from (2.8) that $\mu_{K,Q}$ minimizes the functional

$$\mu \rightarrow E^*(\mu) + \int_K (Q - u_0) d\mu$$

and the minimal value is $E(V_{K,Q})$. Thus $\Lambda^*(\mu) \geq 0$ and $\Lambda^*(\mu) = 0$ precisely for $\mu = \mu_{K,Q}$. In this way one obtains the LDP with rate function arising from the functional E^* in a very natural manner as this functional is itself a Legendre-type transform (see [3]). An advantage of using Proposition 5.6 is that one automatically gets *convexity* of the rate functional. Here, the asymptotic weighted Fekete result Theorem 2.4 is not needed, but one clearly uses the differentiability result Theorem 2.1. Finally, to deduce the equalities

$$-\log \overline{J}(\mu) = -\log \underline{J}(\mu) = E^*(\mu) - \int_K u_0 d\mu - E(V_T)$$

and

$$-\log \overline{J}^Q(\mu) = -\log \underline{J}^Q(\mu) = E^*(\mu) + \int_K (Q - u_0) d\mu - E(V_T),$$

one can appeal to the Rumely-type formula (2.6). Alternately, starting with a strong Bernstein-Markov measure ν and the corresponding sequence of probability measures $\{\sigma_k\}$ on $\mathcal{M}(K)$ in (5.1), the Gärtner-Ellis approach gives the LDP with rate function Λ^* in (5.11) using Proposition 3.1 and the differentiability result Theorem 2.1. Then the *uniqueness* of the rate function gives, as mentioned in Remark 5.5, that this rate function Λ^* is given by \mathcal{I} in (5.8). Thus one obtains the equalities above relating E^* , J and J^Q .

The differentiability result Theorem 2.1 is crucial in either proof of the LDP. In the Angelesco ensemble case studied in [8] as well as more general univariate vector energy settings, a natural formulation of an analogue of this result is unclear. However, in these settings, the analogue of the asymptotic weighted Fekete result Theorem 2.4 is straightforward. We illustrate the ease in obtaining an LDP in the next section in the classical weighted potential-theoretic setting in one variable.

6. The univariate situation.

Let $K \subset \mathbb{C}$ be compact and non-polar. For $\mu \in \mathcal{M}(K)$, we write

$$p_\mu(z) := \int_K \log \frac{1}{|z - \zeta|} d\mu(\zeta),$$

the logarithmic potential function of μ ;

$$I(\mu) := \int_K \int_K \log \frac{1}{|z - \zeta|} d\mu(\zeta) d\mu(z),$$

the logarithmic energy of μ ; and, if $Q \in \mathcal{A}(K)$,

$$I^Q(\mu) := I(\mu) + 2 \int_K Q d\mu$$

is the weighted logarithmic energy of μ with respect to the weight Q . The measure $\mu_{K,Q} = dd^c V_{K,Q}^*$ satisfies $\inf_{\nu \in \mathcal{M}(K)} I^Q(\nu) = I^Q(\mu_{K,Q})$. For compact sets and admissible weights in \mathbb{C} , (2.5) and Theorem 2.4 are standard facts, requiring no more effort than the classical unweighted case (cf., [20]). The version of Corollary 3.2 proved in [12] follows immediately from the univariate version of Proposition 3.1. Theorem 2.4 immediately yields Proposition 4.3 in this setting. In addition, the version of the domination principle used in the proof of Theorem 4.4 in the univariate situation is Theorem 3.2 in Chapter II of [20].

Thus, a priori, the only nonstandard ingredient required to prove Theorems 4.4, 4.6 and 5.1 in \mathbb{C} is (4.9). We show this result also follows from standard potential-theoretic arguments. We begin with $\mu \in \mathcal{M}(K)$ having $I(\mu) < +\infty$ and we let $u(z) := -p_\mu(z)$. As in the proof of Theorem 4.4, the problem is that u is only usc and thus μ is not necessarily of the form $\mu_{K,Q}$ for $Q \in \mathcal{A}(K)$. Proceeding as in Theorem 4.4, we take a sequence of continuous admissible weights $\{Q_j\}$ with $Q_j \downarrow u$ on K . By the domination principle, we get

$$(6.1) \quad u_j := V_{K,Q_j}^* \downarrow u = -p_\mu \text{ on } \mathbb{C}.$$

We have $\mu_j := \mu_{K,Q_j} \rightarrow \mu$ weak-*. Note that $dd^c u_j = -dd^c p_{\mu_j} = \mu_j$ implies

$$(6.2) \quad p_{\mu_j} + u_j = F_j \text{ on } \mathbb{C}$$

for some constant F_j . Finally, for $z \notin K$, the function $\zeta \rightarrow \log |z - \zeta|$ is continuous on K and thus $\mu_j \rightarrow \mu$ weak-* implies that

$$(6.3) \quad \lim_{j \rightarrow \infty} p_{\mu_j}(z) = p_\mu(z) \text{ for } z \in \mathbb{C} \setminus K.$$

Proposition 6.1. *With the preceding notation,*

$$\lim_{j \rightarrow \infty} F_j = 0; \quad \lim_{j \rightarrow \infty} I(\mu_j) = I(\mu); \quad \text{and} \quad \lim_{j \rightarrow \infty} \int_K u_j d\mu_j = \int_K u d\mu.$$

Proof. That $\lim_{j \rightarrow \infty} F_j = 0$ follows immediately from (6.1), (6.2) and (6.3). Noting that $I(\mu_j) = \int_K p_{\mu_j} d\mu_j$, this fact and (6.2) imply that the conditions

$$\lim_{j \rightarrow \infty} I(\mu_j) = I(\mu) \text{ and } \lim_{j \rightarrow \infty} \int_K u_j d\mu_j = \int_K u d\mu$$

are equivalent.

Since $u_{j+1} \leq u_j$, for all $\nu \in \mathcal{M}(K)$

$$I(\nu) + 2 \int_K u_{j+1} d\nu \leq I(\nu) + 2 \int_K u_j d\nu; \text{ thus } I^{Q_{j+1}}(\mu_{j+1}) \leq I^{Q_j}(\mu_j)$$

and $\lim_{j \rightarrow \infty} I^{Q_j}(\mu_j) = \lim_{j \rightarrow \infty} [I(\mu_j) + 2 \int_K u_j d\mu_j]$ exists.

We claim that for all $\nu \in \mathcal{M}(K)$

$$(6.4) \quad I(\nu) + 2 \int_K u d\nu \geq I(\mu) + 2 \int_K u d\mu.$$

Clearly we need only show this for ν with $I(\nu) < +\infty$. Since $I(\mu) < +\infty$, and $\mu - \nu$ is a signed measure of total mass 0 and of compact support, $I(\mu - \nu) \geq 0$ (cf., Lemma I.1.8 in [20]). Using

$$- \int_K u d\nu = \int_K p_\mu d\nu = \int_K p_\nu d\mu \text{ and } I(\mu) = \int_K p_\mu d\mu = - \int_K u d\mu$$

we have

$$0 \leq I(\mu - \nu) = I(\mu) + I(\nu) - 2 \int_K p_\mu d\nu = I(\mu) + I(\nu) + 2 \int_K u d\nu$$

so that

$$I(\nu) + 2 \int_K u d\nu \geq -I(\mu) = I(\mu) + 2 \int_K u d\mu$$

as desired.

Since $u_j \geq u$,

$$(6.5) \quad I(\mu_j) + 2 \int_K u_j d\mu_j \geq I(\mu_j) + 2 \int_K u d\mu_j \geq I(\mu) + 2 \int_K u d\mu$$

where the last inequality comes from taking $\nu = \mu_j$ in (6.4).

We have $I^{Q_j}(\mu_j) \leq I^{Q_j}(\mu)$ (since μ_j minimizes this weighted energy); by monotone convergence using (6.1),

$$I(\mu) + 2 \int_K u d\mu = \lim_{j \rightarrow \infty} [I(\mu) + 2 \int_K u_j d\mu] \geq \lim_{j \rightarrow \infty} [I(\mu_j) + 2 \int_K u_j d\mu_j].$$

Combining this with (6.5), we have

$$(6.6) \quad I(\mu) + 2 \int_K u d\mu = \lim_{j \rightarrow \infty} [I(\mu) + 2 \int_K u_j d\mu] = \lim_{j \rightarrow \infty} [I(\mu_j) + 2 \int_K u_j d\mu_j].$$

We already know that

$$(6.7) \quad \lim_{j \rightarrow \infty} F_j = \lim_{j \rightarrow \infty} [I(\mu_j) + \int_K u_j d\mu_j] = 0;$$

subtracting (6.7) from (6.6) gives the result:

$$\lim_{j \rightarrow \infty} \int_K u_j d\mu_j = I(\mu) + 2 \int_K u d\mu = \int_K u d\mu.$$

□

As in [10], our functionals $J(\mu), W(\mu)$ are related to $I(\mu)$.

Proposition 6.2. *Let $\mu \in \mathcal{M}(K)$. Then*

$$(6.8) \quad -\log W(\mu) = -\log J(\mu) = \frac{1}{2}I(\mu).$$

Proof. Observe that if $I(\mu) < +\infty$, the logarithmic potential function p_μ satisfies $u := -p_\mu \in L(\mathbb{C})$ with $dd^c u = \mu$ and $\int_K u d\mu > -\infty$. In particular, if $\mu = \mu_{K,Q}$ for $Q \in C(K)$, (4.7) and Theorem III.1.3 of [20] show that

$$-\log W(\mu_{K,Q}) = \frac{1}{2}I(\mu_{K,Q}) < +\infty.$$

For $\mu \in \mathcal{M}(K)$ with $I(\mu) < +\infty$, as in the proof of Theorem 4.4, there exists a sequence of continuous admissible weights $\{Q_j\}$ with $Q_j \downarrow -p_\mu$ on K , $\mu_{K,Q_j} \rightarrow \mu$ weak-* and

$$\lim_{j \rightarrow \infty} W(\mu_{K,Q_j}) = W(\mu)$$

Proposition 6.1 gives (6.8) in this case. It remains to show that

$$(6.9) \quad I(\mu) = +\infty \text{ implies } W(\mu) = 0.$$

The proof of Theorem 4.1 (a) of [10] that $\log W(\mu) \leq -\frac{1}{2}I(\mu)$ follows directly from the definition of these functionals and use of $(\zeta, z) \rightarrow \log|z - \zeta|$. Thus (6.9) holds.

□

Corollary 6.3. *Let $Q \in C(K)$. For $\mu \in \mathcal{M}(K)$*

$$(6.10) \quad -\log W^Q(\mu) = -\log J^Q(\mu) = \frac{1}{2}I(\mu) + \int_K Q d\mu = \frac{1}{2}I^Q(\mu).$$

Note that the rate function is

$$\mathcal{I}_{K,Q}(\mu) = \frac{1}{2}[\log I^Q(\mu) - \log I^Q(\mu_{K,Q})]$$

and the speed is $2k^2$.

Remark 6.4. The GUE has joint probability distribution $Prob_k$ on \mathbb{R}^{k+1} given by

$$Prob_k(A) := \frac{1}{Z_k} \cdot \int_A |VDM_k^Q(x_1, \dots, x_{k+1})|^2 \cdot dx_1 \cdots dx_{k+1}$$

for a Borel set $A \subset \mathbb{R}^{k+1}$ where $Q(x) = x^2$ and

$$Z_k = \int_{\mathbb{R}^{k+1}} |VDM_k^Q(x_1, \dots, x_{k+1})|^2 \cdot dx_1 \cdots dx_{k+1}.$$

Here, $\mu_{\mathbb{R},Q} = \frac{1}{\pi} \sqrt{2-x^2} dx$ on the interval $-\sqrt{2} \leq x \leq \sqrt{2}$.

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